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Spectral duality and distribution of exponents for transfer matrices of block-tridiagonal Hamiltonians

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Abstract

I consider a general block-tridiagonal matrix and the corresponding transfer matrix. By allowing for a complex Bloch parameter in the boundary conditions, the two matrices are related by a spectral duality. As a consequence, I derive some analytic properties of the exponents of the transfer matrix in terms of the eigenvalues of the (non-Hermitian) block matrix. Some of them are the single-matrix analogues of results holding for Lyapunov exponents of an ensemble of block matrices, which occur in models of transport. The counting function of exponents is related to winding numbers of eigenvalues. I discuss some implications of duality for the distribution (real bands and complex arcs) and the dynamics of eigenvalues.

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1. Introduction

A tridiagonal Hermitian matrix whose entries are square matrices of size M is a block-tridiagonal matrix. By denoting the diagonal blocks as $H_i = H_i^\dagger$ and the blocks in the next upper diagonal as L_i, the eigenvalue equation written in the block components of an eigenvector $\underline{u}$ is

$$H_i\underline{u}_i + L_i\underline{u}_{i+1} + L_{i-1}^\dagger\underline{u}_{i-1} = E\underline{u}_i. \tag{1}$$

We may view H_i as Hamiltonian matrices of a chain of subsystems, each with M internal states, sequentially coupled by matrices L_i. We shall only require that $\det L_i \neq 0$. Band matrices are in this class, with non-diagonal blocks being triangular. Random band matrices are studied for quantum chaos and transport, mainly by numerical means [1] or by a mapping on a nonlinear supersymmetric sigma model [2]. The block structure is typical of tight binding models, as in Anderson's model for the transport of a particle in a lattice with impurities [3]. Here, the matrices H_i are the Hamiltonians of isolated slices transverse to some direction and the matrices L_i contain the hopping amplitudes between lattice sites of neighbouring slices. Tridiagonal arrays of random matrices were studied in the context of multichannel scattering in mesoscopic or nuclear physics [4] or as multimatrix models in the large size limit [5]. For recent reviews of applications of random matrices in physics, see [6, 7].

The second-order recursive character of (1) makes it useful to introduce a transfer matrix $T_i(E)$, of size $2M \times 2M$:

$$\begin{pmatrix} \underline{u}_{i+1} \\ \underline{u}_i \end{pmatrix} = \begin{pmatrix} L_i^{-1}(E - H_i) & -L_i^{-1}L_{i-1}^\dagger \\ I & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_i \\ \underline{u}_{i-1} \end{pmatrix}. \tag{2}$$

The block components of the eigenvector are reconstructed by applying a product of transfer matrices to an initial block pair. A length N of the chain corresponds to the transfer matrix $T(E) = T_N(E) \cdots T_1(E)$:

$$\begin{pmatrix} \underline{u}_{N+1} \\ \underline{u}_N \end{pmatrix} = T(E) \begin{pmatrix} \underline{u}_1 \\ \underline{u}_0 \end{pmatrix}. \tag{3}$$

The transfer matrix is the main tool for investigating the boundary properties of the Hamiltonian's eigenvectors, or the transmission matrix of the chain coupled to leads. I developed a theory for transfer matrices of general block-tridiagonal Hamiltonians [8, 9]. Recently the formalism was applied to transport in nanotubes and molecules, and generalized to allow for non-invertible off-diagonal blocks [10–13]. In this paper I describe some interesting consequences of a nice algebraic identity involving the characteristic polynomials of the two matrices [8]. Though this spectral duality holds in general, here I restrict to Hermitian block-tridiagonal matrices because of their relevance in physics.

For a chain of length N we must provide boundary conditions. With periodic boundary conditions we require $L_0 = L_N$ for Hermiticity and $\underline{u}_0 = \underline{u}_N, \underline{u}_{N+1} = \underline{u}_1$. However, it turns out to be very convenient to allow for a complex Bloch parameter

$$\underline{u}_0 = \frac{1}{z}\underline{u}_N \qquad \underline{u}_{N+1} = z\underline{u}_1. \tag{4}$$

Therefore, the Hamiltonian matrix is block-tridiagonal with corners

$$\mathcal{H}(z) = \begin{pmatrix} H_1 & L_1 & & & & \frac{1}{z}L_N^\dagger \\ L_1^\dagger & H_2 & L_2 & & & \\ & L_2^\dagger & \ldots & \ldots & & \\ & & & \ldots & \ldots & L_{N-1} \\ zL_N & & & & L_{N-1}^\dagger & H_N \end{pmatrix}. \tag{5}$$

It is Hermitian only for $|z| = 1$. Boundary conditions with $z = \mathrm{e}^{\mathrm{i}\varphi}$ arise when decomposing the eigenproblem (1) for an infinite periodic chain of period N in the eigenspaces of the N-block translation operator, as well as in the topology of an N-site ring with a magnetic flux through it. In general, it is

$$\mathcal{H}(z)^\dagger = \mathcal{H}(1/z^*). \tag{6}$$

The non-Hermitian tridiagonal matrices ($M = 1$), with $z = \mathrm{e}^{Ng}$ and $g \geqslant 0$, were introduced by Hatano and Nelson [14] in a study on vortex depinning in superconductors which promoted a burst of research, see, for example, [15–18]. Spectral properties were analysed in greater detail by Goldsheid and Khoruzhenko [19] who proved that, for $g > g_{cr}$, eigenvalues corresponding to extended states begin to migrate in the complex plane and distribute along the level curve of the single Lyapunov exponent of the model, $\gamma(E) = g$. The two wings of real eigenvalues correspond to exponentially localized eigenvectors, which are insensitive to the boundary.

These features also appear in the more difficult case of block matrices. For $|z|$ sufficiently greater or smaller than unity, the block matrix develops complex eigenvalues which are seen numerically to distribute along lines. The eigenvalues of an eptadiagonal matrix ($M = 3$) with diagonal disorder and unit hopping amplitudes are shown in figure 2.

It is intuitive that there is a connection between boundary properties of eigenvectors, which are described by the transfer matrix, and the response of energy eigenvalues to variations of boundary conditions, namely, the Landauer and Thouless approaches to transport. This intuition has a formulation in the spectral duality. After a short review of duality (section 2), I derive several analytic properties of the exponents of a single transfer matrix (section 3). In particular, I evaluate the counting function of exponents as the winding number of trajectories of eigenvalues of the source block matrix. In section 4, I use duality to describe qualitatively the distribution and dynamics of eigenvalues of Hermitian and non-Hermitian block-tridiagonal matrices.

2. Spectral duality

In this section I review some basic properties of a transfer matrix $T(E)$ of a chain of length N, with $L_0 = L_N$ [8, 9].

$T(E)$ is a matrix polynomial in E of degree N, with a nonzero determinant independent of E and diagonal blocks H_i:

$$\det T(E) = \prod_{i=1}^{N} \frac{\det L_i^\dagger}{\det L_i}. \tag{7}$$

The block structure of the Hamiltonian and the corresponding factorization of the transfer matrix, makes a typical property of transfer matrices apparent.

Proposition 1. *The symplectic property*

$$T(E^*)^\dagger \Sigma_N T(E) = \Sigma_N \qquad \Sigma_N = \begin{pmatrix} 0 & -L_N^\dagger \\ L_N & 0 \end{pmatrix}. \tag{8}$$

Proof. It is a consequence of $L_0 = L_N$ and of the factorization of $T(E)$ into a product of matrices $T_k(E)$, whose inverse is $T_k(E)^{-1} = \Sigma_{k-1}^{-1} T_k(E^*)^\dagger \Sigma_k$. □

Corollary. *If z is an eigenvalue of $T(E)$, then $1/z^*$ is an eigenvalue of $T(E^*)$. For real E, both z and $1/z^*$ are in the spectrum of $T(E)$. If $T(E^*) = T(E)^*$, then both z and $1/z$ are in the spectrum of $T(E)$.*

These statements summarize in the useful identity

$$\det[T(E) - z] = z^{2M} \det T(0) \det[T(E^*) - 1/z^*]^*. \tag{9}$$

Let $\underline{u}$, with blocks $\underline{u}_1, \ldots, \underline{u}_N$, be an eigenvector of $\mathcal{H}(z)$ with eigenvalue E. Then, by equation (3) and after imposing the boundary conditions (4),

$$\begin{pmatrix} z\underline{u}_1 \\ \underline{u}_N \end{pmatrix} = T(E) \begin{pmatrix} \underline{u}_1 \\ 1/z\underline{u}_N \end{pmatrix} \tag{10}$$

which means that z is an eigenvalue of $T(E)$ with an eigenvector of blocks $z\underline{u}_1$ and $\underline{u}_N$. However, the converse is true: given an eigenvector of $T(E)$ with eigenvalue z one reconstructs, via products of matrices $T_k(E)$, the whole eigenvector of $\mathcal{H}(z)$ with eigenvalue E. Therefore, $\det[E - \mathcal{H}(z)] = 0$ if and only if $\det[T(E) - z] = 0$; this duality among eigenvalues is made precise as an identity among characteristic polynomials [8].

Proposition 2. *The spectral duality*

$$\det[E - \mathcal{H}(z)] = (-z)^{-M} \det[L_1 L_2 \ldots L_N] \det[T(E) - z]. \tag{11}$$

Proof. We must show that $\det[T(E) - z]$ is a polynomial in E of degree NM with leading coefficient $(-z)^M \det[L_1 \cdots L_N]^{-1}$. To this end, we first consider the leading terms in E of both sides of equation (9): the equality implies that the leading term of $\det[T(E) - z]$ is proportional to z^M. Next we derive the following leading block stucture of $T(E)$:

$$T(E) \approx \begin{pmatrix} E^N Q & -E^{N-1} Q L_N^\dagger \\ E^{N-1} L_N Q & -E^{N-2} L_N Q L_N^\dagger \end{pmatrix} \qquad Q = (L_1 \ldots L_N)^{-1}.$$

The leading term in E of $\det[T(E) - zI]$ with the constraint of being proportional to z^M, is provided by the diagonal factors $\det(E^N Q)$ and $\det(-zI)$. □

Corollary. *If Im* $E \neq 0$ *then* $T(E)$ *has no eigenvalues on the unit circle.*

Proof. For $|z| = 1$ the matrix $\mathcal{H}(z)$ is Hermitian, and for $\mathrm{Im}\, E \neq 0$ it is always $\det[E - \mathcal{H}(z)] \neq 0$. By duality, this implies $\det[T(E) - z] \neq 0$. □

Notes. In [9] I provided a representation of $T(E) - z$ in terms of the corner blocks of the resolvent of $\mathcal{H}(z)$. I also stated the spectral duality for the matrix $T^\dagger T$.
In the tridiagonal case ($M = 1$), blocks are numbers. If $\lambda = L_1, \ldots, L_N$ the spectral duality simplifies to the known expression

$$\det[E - \mathcal{H}(z)] = \lambda \,\mathrm{Tr}\, T(E) - \lambda z - \lambda^* \frac{1}{z}. \tag{12}$$

3. The spectrum of exponents

Let us fix E real or complex and denote as $z_a(E) = \mathrm{e}^{\xi_a + \mathrm{i}\varphi_a}$, $a = 1, \ldots, 2M$, the eigenvalues of $T(E)$. The real numbers $\xi_a(E)$ are the *exponents* of the transfer matrix. For real E the symplectic property (8) assures that exponents come in pairs $\pm\xi_a(E)$.

The property $|\det T(E)| = 1$, see (7), implies

$$\sum_{a=1}^{2M} \xi_a(E) = 0. \tag{13}$$

When considering an ensemble of random Hamiltonian matrices with tridiagonal block structure, one is often interested in the corresponding ensemble of transfer matrices. Being a product of N random matrices, a transfer matrix develops exponents $\xi_a(E)$ that asymptotically grow linearly in the length N [20], with a coefficient known as the *Lyapunov exponent*:

$$\gamma_a(E) = \lim_{N\to\infty} \frac{1}{N} \langle \xi_a(E) \rangle. \tag{14}$$

For tridiagonal random matrices there is just one pair of opposite Lyapunov exponents, which can be evaluated with the Herbert–Jones–Thouless formula [21], with the knowledge of the average density of eigenvalues

$$\gamma(E) = \text{constant} + \int \mathrm{d}E' \rho(E') \log|E - E'|.$$

The extension to a complex value E is discussed in [22]. For the Anderson model [23] or band random matrices [24] there are several numerical studies of Lyapunov spectra. In these cases of great physical interest, transfer matrices are derived from the Hamiltonians and the analytic approach is difficult.

It is interesting to note that an analytic formula relating the distribution of the exponents to the spectrum of the Hamiltonian is possible. Note that the following statements are true for a single and general block-tridiagonal Hamiltonian.

We shall deduce several results from

Proposition 3.

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \log|\det[E - \mathcal{H}(e^{\xi+i\varphi})]| = \sum_{i=1}^{N} \log|\det L_i| + \frac{1}{2} \sum_{a=1}^{2M} |\xi - \xi_a(E)|. \quad (15)$$

Proof. The duality relation (11) gives

$$\begin{aligned} \log|\det[E - \mathcal{H}(e^{\xi+i\varphi})]| - \sum_i \log|\det L_i| &= -M\xi + \frac{1}{2}\sum_{a=1}^{2M} \log|e^{\xi_a+i\varphi_a} - e^{\xi+i\varphi}|^2 \\ &= \frac{1}{2}\sum_{a=1}^{2M} \{\xi_a + \log[2\cosh(\xi_a - \xi) - 2\cos(\varphi_a - \varphi)]\} \\ &= \frac{1}{2}\sum_{a=1}^{2M} |\xi_a - \xi| - \sum_{\ell=1}^{\infty}\sum_{a=1}^{2M} \frac{\cos \ell(\varphi_a - \varphi)}{\ell} e^{-\ell|\xi_a - \xi|}. \end{aligned} \quad (16)$$

We used equation (13) and the formula (see equation 1.448.2 in [25])

$$\log[2\cosh x - 2\cos y] = |x| - 2\sum_{\ell=1}^{\infty} \frac{\cos \ell y}{\ell} e^{-\ell|x|}.$$

Equation (16) is the Fourier expansion of $\log|\det[E - \mathcal{H}(e^{\xi+i\varphi})]|$, which is a periodic function of φ. The constant mode is just the proposition. □

In the special case $\xi = 0$, the matrix $\mathcal{H}(e^{i\varphi})$ is Hermitian, and (15) yields a formula for the *sum of positive exponents* (E can be complex).

Proposition 4.

$$\sum_{\xi_a>0} \xi_a(E) = -\sum_i \log|\det L_i| + \int_0^{2\pi} \frac{d\varphi}{2\pi} \log|\det(E - \mathcal{H}(e^{i\varphi}))|. \quad (17)$$

By taking the derivative in ξ of (15) we obtain the *spectral counting function*, which counts the exponents less than ξ, for any complex value E;

$$\mathcal{N}(\xi_a(E) \leqslant \xi) = \sum_{a=1}^{2M} \theta(\xi - \xi_a(E)) \quad (18)$$

$$= M + \frac{d}{d\xi}\int_0^{2\pi} \frac{d\varphi}{2\pi} \log|\det[E - \mathcal{H}(e^{\xi+i\varphi})]|. \quad (19)$$

We now write $|\det[E-\mathcal{H}(z)]|$ in terms of the eigenvalues $E_n(z)$ and their complex conjugate, which equals $E_n(1/z^*)$, and evaluate the derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\log|\det[E-\mathcal{H}(\mathrm{e}^{\xi+\mathrm{i}\varphi})]|\\ &= -\frac{1}{2}\sum_n\left(\frac{1}{E-E_n(z)}\frac{\partial E_n(z)}{\partial\xi}+\frac{1}{E^*-E_n(z)^*}\frac{\partial E_n(z)^*}{\partial\xi}\right)\\ &= -\frac{1}{2\mathrm{i}}\sum_n\left(\frac{1}{E-E_n(z)}\frac{\partial E_n(z)}{\partial\varphi}-\frac{1}{E^*-E_n(z)^*}\frac{\partial E_n(z)^*}{\partial\varphi}\right).\end{aligned} \tag{20}$$

Let us denote by $N_+(E)$ and $N_-(E)$ the numbers of positive and negative exponents of $T(E)$, respectively. We have

Proposition 5. $N_+(E)=N_-(E)$.

Proof. For real E we know that the exponents come in pairs $\pm\xi_a(E)$, because of proposition 1. Let us consider the case $\mathrm{Im}\,E\neq 0$.

As a consequence of duality we derived that no eigenvalue of $T(E)$ is on the unit circle, thus all exponents are nonzero: $N_-+N_+=2M$. Therefore, if we set $\xi=0$ in (19), the left term is N_-. We now show that $N_-=M$ or, equivalently, that the integral in (19) vanishes for $\xi=0$. In expression (20) the eigenvalues $E_n(\mathrm{e}^{\mathrm{i}\varphi})$ are real periodic functions of φ in $[0,2\pi]$, then

$$N_-(E)-M=\sum_{n=1}^{NM}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\frac{\mathrm{d}E_n}{\mathrm{d}\varphi}\frac{\mathrm{Im}\,E}{(\mathrm{Re}\,E-E_n)^2+(\mathrm{Im}\,E)^2}=0. \qquad \square$$

As a function of φ (ξ is fixed) an eigenvalue $E_n(\mathrm{e}^{\xi+\mathrm{i}\varphi})$ makes a loop γ_n in the complex E plane. The loop γ_n^* of E_n^* is specular with respect to the real axis. Integration in φ of (20) yields Cauchy integrals

$$\mathcal{N}(\xi_a(E)<\xi)=M+\frac{1}{2}\sum_n\left(\int_{\gamma_n}\frac{\mathrm{d}E'}{2\pi\mathrm{i}}\frac{1}{E'-E}-\int_{\gamma_n^*}\frac{\mathrm{d}E'}{2\pi\mathrm{i}}\frac{1}{E'-E^*}\right).$$

The first integral is the winding number of the (oriented) trajectory γ_n of $E_n(\mathrm{e}^{\xi+\mathrm{i}\varphi})$ around the value E. The second integral is the winding number of γ_n^* around E^* and has opposite sign because of opposite orientation.

We then obtain a nice geometric result.

Proposition 6. $\mathcal{N}(\xi_a(E)<\xi)=M+\mathcal{W}(E)$**.** *The number of exponents of $T(E)$ less than ξ equals M plus the total winding number of loops of eigenvalues $E_n(\mathrm{e}^{\xi+\mathrm{i}\varphi})$, $-\pi<\varphi<\pi$, that encircle E.*

As I mentioned, these formulae hold for a single general block-tridiagonal Hamiltonian matrix and its transfer matrix. For a statistical ensemble of Hamiltonians one performs ensemble averages in place of a phase average, and deals with Lyapunov exponents. Souillard (quoted in [20]) obtained the following formula for the positive Lyapunov exponents, which is the statistical analogue of (17):

$$\frac{1}{M}\sum_a\gamma_a(E)=\text{constant}+\int\mathrm{d}E'\rho(E')\log|E-E'| \tag{21}$$

where $\rho(E)$ is the ensemble averaged spectral density of the Hermitian Hamiltonians. I am not aware of any statistical analogue of (19).

4. Bands, arcs and energy level motion

Spectral duality provides information on the positions of the eigenvalues of $\mathcal{H}(z)$ and their motion under variations of the boundary parameter z.

It is useful to introduce the notion of a *discriminant*. If $z_a(E)$ is an eigenvalue of $T(E)$, the discriminant is the eigenvalue of $T(E)+T(E)^{-1}$ with the same eigenvector:

$$\Delta_a(E) = z_a(E) + \frac{1}{z_a(E)} = 2\cosh\xi_a\cos\varphi_a + 2\mathrm{i}\sinh\xi_a\sin\varphi_a. \tag{22}$$

Since the symplectic property implies that $1/z_a^*$ is an eigenvalue of $T(E^*)$, it is, in general

$$\Delta_a(E^*) = \Delta_a(E)^*. \tag{23}$$

Let us begin with the simpler case where H_i, L_i are real matrices. Then $T(E^*) = T(E)^*$ and the eigenvalues of the transfer matrix come in pairs z_a and $1/z_a$, $a = 1,\ldots,M$. Moreover, if E is real, the characteristic polynomial of $T(E)$ has real coefficients and roots also come in pairs z_a, z_a^*.

The spectral duality reads

$$\prod_{a=1}^{M}\left[\left(z_a+\frac{1}{z_a}\right)-\left(z+\frac{1}{z}\right)\right] = \det[L_1\ldots L_n]^{-1}\det[E-\mathcal{H}(z)]. \tag{24}$$

Therefore, the M equations

$$\Delta_a(E) = z + \frac{1}{z} \tag{25}$$

provide the NM eigenvalues of $\mathcal{H}(z)$, which are naturally classified in subsets with label a. We consider two cases: $|z| = 1$ and z real.

When $z = \mathrm{e}^{\mathrm{i}\varphi}$ all eigenvalues of $\mathcal{H}(\mathrm{e}^{\mathrm{i}\varphi})$ are real periodic functions of $-\pi \leqslant \varphi < \pi$. Each equation $\Delta_a(E) = 2\cos\varphi$ provides a number $n_a \geqslant 0$ of real solutions, and $n_1+\cdots+n_M = NM$. This means that $y = \Delta_a(E)$, as a function of the real variable E, crosses n_a times the strip $-2 \leqslant y \leqslant 2$ parallel to the E-axis. No extrema are allowed in the strip, for all branches must cut n_a times the lines $y = 2\cos\varphi$, to ensure that $\mathcal{H}(\mathrm{e}^{\mathrm{i}\varphi})$ has NM real eigenvalues for all φ. All branches of the functions $\Delta_a(E)$ cross their bands in the eigenvalues of $\mathcal{H}(i)$ (see figure 1).

By changing φ the eigenvalues span bands in the real axis

$$E = E_{a,j}(\varphi) \qquad a = 1,\ldots,M \qquad j = 1,\ldots,n_a$$

with extrema given by the eigenvalues of the periodic ($\varphi = 0$) and antiperiodic ($\varphi = \pi$) Hamiltonians. The velociy of level motion is

$$\frac{\partial E}{\partial\varphi} = -\frac{2\sin\varphi}{\Delta_a'(E)}. \tag{26}$$

In the turning points the velocity vanishes. The second derivative is known as the curvature [1, 26, 27]. This dynamics is invariant under the 'time-reversal' operation $\varphi \to -\varphi$, corresponding to the transposition of the Hamiltonian matrix. Bands with the same label a may at most share an extremum, while bands related to different a may overlap. When a branch of Δ_a and a branch of $\Delta_{a'}$ cross inside the strip at a point $(E, 2\cos\varphi)$, there is a crossing of two eigenvalues of $\mathcal{H}(\mathrm{e}^{\mathrm{i}\varphi})$ (and a collision of two pairs of eigenvalues of $T(E)$ in the unit circle). This is a highly non-generic occurrence for one-parameter Hermitian matrices. Energy bands of Hermitian periodic tridiagonal matrices ($M = 1$) were studied in [28].

When $z = \mathrm{e}^{Ng}$, with $g \geqslant 0$, the matrix $\mathcal{H}(z)$ is real. The equations $\Delta_a(E) = 2\cosh(Ng)$ provide all NM eigenvalues in subsets labelled with a. They imply the equation $\xi_a(E) = \pm Ng$.

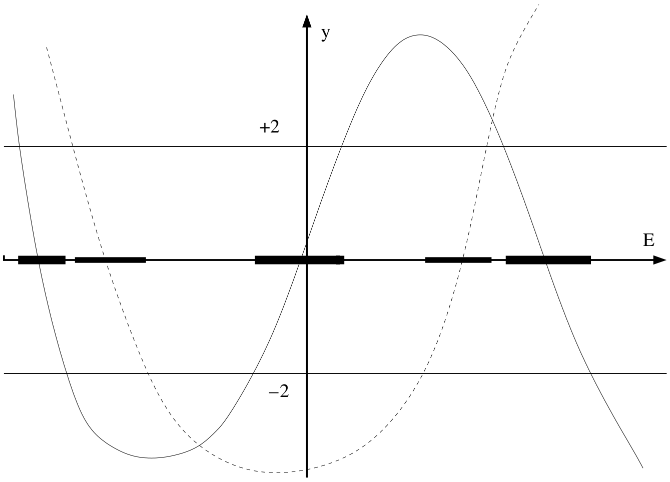

Figure 1. Energy bands of a pentadiagonal matrix ($M = 2$), solution to $\Delta_a(E) = 2\cos\varphi$, $a = 1, 2$, $-\pi \leqslant \varphi < \pi$.

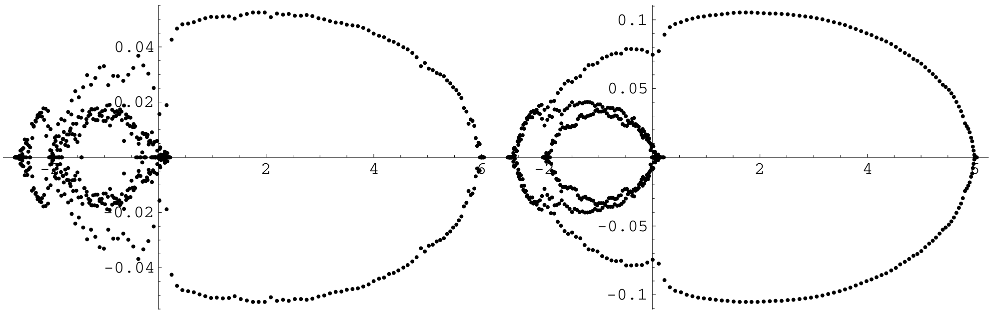

Figure 2. The 600 eigenvalues of an heptadiagonal matrix ($M = 3$, $N = 200$), with diagonal uniform disorder $|a_i| < 0.5$, unit off-diagonal elements, and $z = 20$ (left), $z = 390$ (right).

For finite g and large N, given that the exponents grow linearly in N with a coefficient that defines the Lyapunov exponent, the eigenvalues of the non-Hermitian matrix $\mathcal{H}(z)$ distribute along M level lines

$$\gamma_a(E) = g. \tag{27}$$

The distribution along arcs is shown in figure 2, for large z. For small z the pattern of the eigenvalue distribution is complicated and intertwined. For g small enough, in continuity with the description given for the periodic case, the eigenvalues still belong to the real axis, outside their periodicity bands. By increasing g, eigenvalues with the same label a approach pairwise until a pair condenses. Correspondingly, an extremum is reached for the function $\Delta_a(E)$ and the pair of eigenvalues acquire opposite imaginary parts. There is a critical value g_a of g for this to happen for each label a:

$$\Delta_a'(E_a) = 0 \qquad \Delta_a(E_a) = 2\cosh(Ng_a). \tag{28}$$

Finally, let us mention the case where $H_i = H_i^\dagger$ and L_i are not real matrices. Another spectral identity is needed, which follows from spectral duality and the symplectic property [8]. It holds for any value of z or E in the complex plane

$$\det\left[T(E)+T(E)^{-1}-\left(z+\frac{1}{z}\right)\right]=|\det L_1\dots L_n|^{-2}\det[E-\mathcal{H}(z)]\det\left[E-\mathcal{H}\left(\frac{1}{z}\right)\right]. \qquad (29)$$

For $z = e^{i\varphi}$ the right-hand side is zero in $2NM$ real solutions, $E_i(\varphi)$ and $E_i(-\varphi), i = 1, \dots, NM$, which span the same NM bands as φ varies in $-\pi, \pi$. They are degenerate in $\varphi = 0$ $(y = 2)$ and $\varphi = \pi$ $(y = -2)$. The strip $-2 \leqslant y \leqslant 2$ is thus crossed by $2NM$ branches $y = \Delta_a(E)$ which join pairwise at the boundaries $y = \pm 2$ of the strip. Each pair, when intersected with the line $y = 2\cos\varphi$, determines the same band which the eigenvalue $E(\varphi)$ covers with different speeds in the two directions.

For $z = \pm e^{Ng}$ we have again condensation of pairs of eigenvalues, which no longer become complex conjugate pairs. For large N the eigenvalues move into M level curves $\gamma_a(E) = g$ (the case $g < 0$ leads to a different set of curves since in this case Lyapunov exponents need not be opposite pairs).

5. Conclusion

The spectral duality is a simple identity that links the eigenvalues of a matrix with block-tridiagonal structure to those of the corresponding transfer matrix. I have deduced some analytic properties for the exponents (sum of exponents, $N_+ = N_-$, counting function). They involve a phase average on eigenvalues of the block matrix.

The large N stability of exponents allows us to describe the distribution of complex eigenvalues of the block matrix along arcs. The discriminants classify real eigenvalues (periodic case) in bands, and govern their motion and collisions. These exact properties are expected to allow a more analytic approach to the difficult study of Lyapunov spectra of transfer matrices, which are *derived* from an ensemble of random Hamiltonians. They also extend to block matrices some results which were known for purely tridiagonal matrices.

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